Radar Imaging

1. Inverse Problem

A general system function can be represented by both in time and frequency domain. Their relation is defined by Fourier transformation.

When parameters of a system are time invariable, the input $x(t)$ and the output $y(t)$ of the system is related by an impulse response of the system function $h(t)$ by:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) \otimes h(t) \qquad (1.1)$$

The operator $\otimes$, which is defined in the time domain, is convolution. The spectrum of $x(t)$, $y(t)$ and $h(t)$ are given as $X(\omega), Y(\omega)$ and $H(\omega)$, and they are related in the frequency domain by:

$$Y(\omega) = H(\omega) \cdot X(\omega) \qquad (1.2)$$

where $H(\omega)$ is the system function, or the transfer function of a system..

Now we think about a simple inverse problem. We assume that we know the system function, and output was measured. The inverse problem is estimation of the input $x(t)$ by using the known parameters $h(t)$ and $y(t)$. This kind of problem is normally called "deconvolution", because it is the reverse operation of convolution. This kind of problem can be found in signal forming in radar. The received radar signal is suffered from system function such as phase delay of antennas, therefore the received signal have to be processed to recover the original signal.

1) Inverse filter

Theoretically, this problem can be solved exactly by the inverse filter. By using (1.2) we can obtain the Fourier spectrum of the input as:

$$X(\omega) = \frac{Y(\omega)}{H(\omega)} = Y(\omega) \cdot H^{-1}(\omega) \qquad (1.3)$$

Thus, by taking the inverse Fourier transformation of (1.3) we have

$$x(t) = F^{-1}\left[\frac{Y(\omega)}{H(\omega)}\right] \qquad (1.4)$$

However, the inverse filter is very strongly affected by noise.

Assume that the output signal contains noise $n(t)$, which is incoherent with the input signal $x(t)$.

$$y(t) = x(t) + n(t) \qquad (1.5)$$

By substituting (1.5) into (1.4) we have:

$$x(t) \to F^{-1}\left[\frac{Y(\omega)}{H(\omega)} + \frac{N(\omega)}{H(\omega)}\right] \qquad (1.6)$$

where $\to$ indicates the estimator. The fist term is the original estimator of the input, but the second term is generated from the noise. When the signal to noise ratio (SNR) is great enough at all the frequency range, there is no problem, but when SNR is poor at some frequency range, the second term becomes large. For instance, if the $H(\omega)$ is a band-pass filter, at the frequency range of the rejected frequency range, SNR is very low, because $X(\omega)$ is small. In this frequency range, $H(\omega) \cong 0$ due to the band-pass characteristics. Therefore the denominator of the second term in (1.6) is zero and the second term diverges.

This problem can be solved by using an additional band-pass filter $G(\omega)$. If we set $G(\omega)$ as a band-pass filter having the frequency range inside the spectrum of the input $X(\omega)$,

$$x(t) \to F^{-1}\left[\frac{Y(\omega)\cdot G(\omega)}{H(\omega)} + \frac{N(\omega)\cdot G(\omega)}{H(\omega)}\right] \cong F^{-1}\left[\frac{Y(\omega)\cdot G(\omega)}{H(\omega)}\right] = x(t) \otimes g(t) \qquad (1.7)$$

We have to notice that the estimated signal is very strong affected by the impulse response of the additional band-pass filter $g(t)$. The convolution of $x(t)$ and $g(t)$ normally cause artifact and broaden the signal. In order to reduce the artifact, $g(t)$ is normally a smooth function such as a Gussian pulse, so that the frequency range is relatively narrow and does not cause oscillation in the time domain.

2) Matched Filter

More common technique to solve the problem of the noise is the matched filter. The matched filter is defined as:

$$x(t) \to \int_{-\infty}^{\infty} y(\tau)h(\tau - t)d\tau \qquad (1.8)$$

Mathematically, this is a cross-correlation of $y(t)$ and $h(t)$. The evaluation of (1.8) can be given by Fourier transformation as:

$$x(t) \to F^{-1}\left[Y(\omega)\cdot H^{*}(\omega)\right] \qquad (1.9)$$

where * denotes the complex conjugate of the function $H(\omega)$. The matched filter is known as a stable estimate even the signal has low SNR. However, when SNR is good enough, the matched filter gives the result having lower resolution compared to the result obtained by the inverse filter.

3) Minimum distance

The same problem can be estimated by a different approach. If we assume a model of the $x(t)$ as $x_m(t)$, then the output corresponding to the model can be calculated as:

$$y_m(t) = h(t) \otimes x_m(t) \qquad (1.10)$$

Then the difference of the output of the model and the measured output can be formulated as:

$$\varepsilon(x_m(t)) = \int_{-\infty}^{\infty} \left| x(t) - x_m(t) \right|^2 dt \qquad (1.11)$$

If this residual is minimized, we can conclude that the model $x_m(t)$ is the good estimate of the real input $x(t)$. In order to minimize the residual, we have to change the model $x_m(t)$ and evaluate the residual to find the minimum.

2. Range Imaging

Radar is a technique to estimate the location and the shape of the radar target by electromagnetic wave. This situation can be modeled by the system function as $h(t)$ as transmitting signal, $y(t)$ as received radar signal and $x(t)$ as an impulse response of a radar system under measurement. This system can be modeled as:

$$x(t) = r_i \delta(t - t_i) \qquad (2.1)$$

where r_i is the reflectivity of the radar target and $t_i = 2d_i / v$ is the two-way travel time from the target. Our task is estimation of r_i and t_i from the measured $y(t)$, when the input signal $h(t)$ is not an ideal impulse, so the measured $y(t)$ is a train of a finite duration pulse.

Inverse filtering or the matched filtering can be used to recover the impulse train of $x(t)$, and is the estimate of the radar target parameters. In this case, the transmitting signal waveform $h(t)$ must be known. This signal is normally called the "reference signal", and it must be also determined from the measurement. For example, the reference signal can be obtained by averaging many received signals.

3. 2-D Inverse problem

Now we extend the inverse problem into 2-D case. Although the 2-D inverse filtering can also be used, matched filter is more common, due to its stability. 2-D matched filter can be defined by: Now the input function is determined by $u(x, y)$ and the system function as $h(x, y)$ and the output as $f(x, y)$. The relation between these function is written by:

The correlation function is defined by

$$f(x,y)=\iint u(x-x',y-y')\cdot h(x',y')dx'dy'=u(x,y)\otimes_x\otimes_y h(x,y) \qquad (3.1)$$

The matched filter to estimate $u(x,y)$ from the measured $f(x,y)$ and the known system function $h(x,y)$ can be given by:

$$\begin{aligned} u(x,y) &\to \iint h(x'-x,y'-y)\cdot f(x',y')dx'dy' \\ &= F^{-1}\left[H^*(k_x,k_y)\cdot F(k_x,k_y)\right] \end{aligned} \qquad (3.2)$$

This is the 2-D correlation between $h(x,y)$ and $f(x,y)$.

4. Radar Imaging

4.1 Scattering Model from Radar Targets

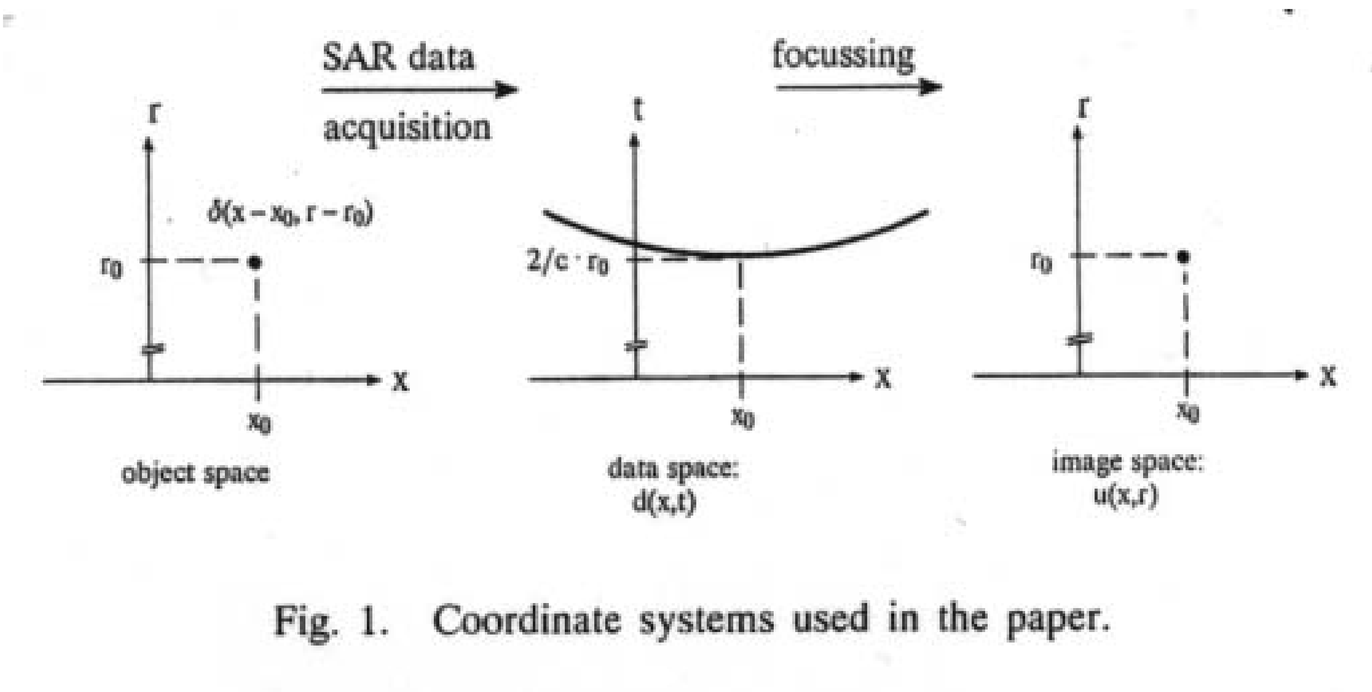

Fig.4.1 1-D scanning case for 2-D imaging radar

At first, we think a case of one-dimensional scanning along am axis x. The measured radar signal is $d(x,y,t)$ and the focused (processed) signal is $u(x,y)$, where y is a range axis, which is perpendicular to the x axis. A point scatter $\sigma(x',y')$ is located. The scatter can be modeled as:

$$\sigma(x',y')=\sum \sigma_i\delta(x'-x_{0i},y'-y_{0i})$$

where (x_{0i},y_{0i}) is the location of the *i*-th point target. Note that range axis y is normally denoted by r in SAR textbooks as shown in Fig.4.1.

The response to the point scatter is:

$$
\begin{aligned}
d(x,y,t)|_{y=0} &= \iint \sigma(x',y')h(t-\frac{2R(x'-x,y'-y)}{c})dx'dy' \\
&= \iint \sum \sigma_{0i}\delta(x'-x_{0i},y'-y_{0i})h(t-\frac{2R(x'-x,y'-y)}{c})dx'dy' \\
&= \sum \sigma_{0i}h(t-\frac{2\sqrt{(x_{0i}-x)^2+(y_{0i}-y)^2}}{c})\ |_{y=0}
\end{aligned}
\qquad (4.1)
$$

where $R(x,y)=\sqrt{x^2+y^2}$ (4.2)

and $h(t)$ is the transmitted signal.

4.2 Radar Imaging in Time-Domain

The estimate of the radar reflectivity is given by correlation of the measured signal $d(x,t)$ and the response of a single point scatter $h(t)$ and is given by:

$$
\sigma(x,y)\rightarrow u(x,y)=\iint d(x',t)\cdot h(t-\frac{2R(x'-x,y'-y)}{c})dx'dt \qquad (4.3)
$$

where d_j indicates the measured data acquired at $x'=x_j$. This is the algorithm to obtain the radar image in the time domain. This is the output of the matched-filter. It should be noted that the integration of (4.3) does not have the form of convolution, shown in (3.1), therefore a technique of FFT cannot be used in this form.

4.3 Diffraction Stacking

If we assume the signal form $h(t)$ as an dirac-impulse $\delta(t)$, (4.3) can be simplified and we have the following formulation representing "Diffraction Stack" which is commonly used for seismic signal processing.

$$
\begin{aligned}
\sigma(x,y)\rightarrow u(x,y) &= \iint d(x',t)\cdot h(t-\frac{2R(x'-x,y'-y)}{c})\ |_{y'=0}\ dx'dt \\
&\cong \iint d(x',t)\cdot \delta(t-\frac{2R(x'-x,y'-y)}{c})\ |_{y'=0}\ dx'dt \\
&= \int d(x',t=\frac{2R(x'-x,y'-y)}{c})\ |_{y'=0}\ dx' \\
&= \int d(x',t=\frac{2R(x'-x,y)}{c})dx' \\
&= \sum_j d_j(t=\frac{2R(x'-x,y)}{c})
\end{aligned}
\qquad (4.4)
$$

4.4 f-k migration

By taking the Fourier transformation of (4.3) as for t we have:

$$U(x,y,\omega)=\int D(x,y,\omega)|_{y=0}\, H(\omega)\exp(j\frac{2R}{c}\omega)dx' \qquad (4.5)$$

By using the relation

$$k_y=\sqrt{\frac{4\omega^2}{c^2}-k_x^2} \qquad (4.6)$$

with $\exp(-jk\sqrt{x^2+y^2})=\exp(-jk_x x-jk_y y)$ (4.7)

we can take the Fourier transform as for x then we have:

$$U(k_x,y,\omega)=\frac{1}{2\pi}D(k_x,y,\omega)|_{y=0}\, H(\omega)\exp(jk_x x+jk_y y) \qquad (4.8)$$

This equation is the same as (6.3.6) in "Fundamental of GPR signal interpretation".

$$u(x,y,0)=\iint U(k_x,0,\omega)e^{+jk_y y}e^{+jk_x x}dk_x d\omega \qquad \text{(6.3.6 in Fundamental of GPR signal processing)(4.9)}$$

This approach is normally called as f-k migration. This is a exactly the same process as the Time—Domain imaging, given in (4.3), however, we need interpolation to obtain the spectrum in k_y -domain used in (4.7). The spectrum cannot directly be given by measurement, and this interpolation can cause numerical error, when the data acquisition density is not high enough.

4.5 Radar Imaging by using FFT

We rewrite (4.2) and (4.3)

$$\sigma(x,y)\rightarrow u(x,y)=\iint d(x',t)\cdot h(t-\frac{2R(x'-x,y'-y)}{c})\ \Big|_{y'=0} dx'dt \qquad (4.10)$$

$$R(x,y)=\sqrt{x^2+y^2} \qquad (4.11)$$

The numerical calculation of (4.10) is very time-consuming, and if we can use FFT algorithm, it would be very effective. In order to apply FFT to (4.10), we have to use a few approximations.

At fist, we assume that the radar target is located at a constant position as for y, i.e., $y=y_0$. For most of SAR from satellites and airplanes, y is the height of the radar from the ground surface. The ground surface height variation is very small compared to the height of the radar, then the approximation of

$$y=y_0 \qquad (4.12)$$

can be satisfied for most of the targets. Then (4.10) can be rewritten as:

$$\sigma(x,y)\rightarrow u(x,y)=\iint d(x',t)\cdot h(t-\frac{2R(x'-x,y_0)}{c})dx'dt \qquad (4.13)$$

Then, this formulation can be modified by using convolution and we have:

$$\sigma(x,y) \to u(x,y) = \iint d(x',t) \cdot h(t - \frac{2R(x'-x,y_0)}{c})dx'dt$$
$$= \int d(x',t) \otimes_t h(\frac{2R(x'-x,y_0)}{c})dx' \qquad (4.14)$$
$$= d(x,t) \otimes_t \otimes_x h(x,t)$$

By using 2-D Fourier transformation, we have:

$$U(k_x,k_y) = D(k_x,k_y)H(k_x,k_y) \qquad (4.15)$$

This is the most commonly used algorithm in SAR imaging .

4.6 3-D Radar Imaging

The algorithms for 2-D model can be extended into 3-D imagine case as shown in Fig.4.2.

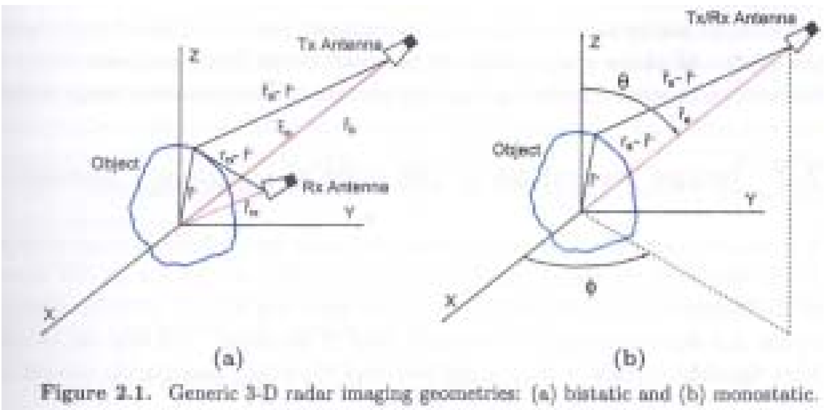

Fig.4.2 3-D radar imaging model

The task of radar imaging is to estimate the reflectivity of $\sigma(r')$ which is distributed at the spatial coordinate of r'. An image function $I(r')$ should be the estimation of $\sigma(r')$.

The model of the received radar signal from the distributed scatters (radar target) can be given as:

$$s_r(t,x,y) = \iiint_V \sigma(r')s\left[t - \frac{2(R_{tx}+R_{rx})}{c}\right]dr' \qquad (4.1)$$

where $s(t)$ is a transmitted signal, $R_{tx}+R_{rx}$ is the one-way range from the transmitter, receiver to the target, and c is the velocity of light. Then the estimation of the target can be given by correlation as:

$$\sigma(r') \to \int_t \iint_A s_r(t,x,y)s\left[t - \frac{2(R_{tx}+R_{rx})}{c}\right]dxdydt \qquad (4.2)$$

Appendix A

(Proof of (1.9)

We define Fourier transformations as:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} Y(\omega)e^{j\omega t}d\omega \\
h(t) &= \int_{-\infty}^{\infty} H(\omega)e^{j\omega t}d\omega \\
Y(\omega) &= \int_{-\infty}^{\infty} y(\omega)e^{-j\omega t}d\omega \\
H(\omega) &= \int_{-\infty}^{\infty} h(\omega)e^{-j\omega t}d\omega
\end{aligned}
\qquad \text{(A.1)}
$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty} y(\tau)h(\tau-t)d\tau = \int_{-\infty}^{\infty} y(\tau+t)h(\tau)d\tau \\
&= \iint Y(\omega)e^{j\omega(\tau+t)}d\omega \cdot h(\tau)d\tau \\
&= \int Y(\omega)\left\{\int h(\tau)e^{j\omega(\tau+t)}d\tau\right\}d\omega \\
&= \int Y(\omega)e^{j\omega t}\left\{\int h(\tau)e^{j\omega\tau}d\tau\right\}d\omega \\
&= \int Y(\omega)H^*(\omega)e^{j\omega t}d\omega
\end{aligned}
$$